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HIGHER-ORDER AVERAGING SCHEMES IN SYSTEMS WITH FAST AND SLOW PHASES†

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Non-linear oscillatory systems containing a fast phase and a relatively slow phase are considered. A modified averaging method is proposed for the situation in which the slow variables, averaged over the fast phase, do not change. A procedure for separating the variables over substantially longer time intervals with respect to a small parameter is proposed and justified; over these extended time intervals, all the variables experience considerable evolution. Examples illustrating the efficiency of the proposed approach are presented. © 2002 Elsevier Science Ltd. All rights reserved.

In non-linear oscillatory systems one frequently encounters situations in which the evolution of the osculating variables occurs at different average rates relative to the powers of some natural small parameter. A variety of problems are described by such systems: in the theory of oscillations of non-linear mechanical systems (oscillators and pendulums), the dynamics of rigid bodies and gyroscopes, orbital motions and rotations of natural and artificial celestial bodies, the problem of non-linear standing waves in a liquid (parametric resonance), etc. It is of considerable interest, from both theoretical and applied points of view, to investigate the evolution of a system over a time interval long enough to allow a substantial variation of the osculating variables, including the slowest ones, to occur.

It turns out that for many important cases one can apply and justify a modified scheme of the Krylov–Bogolyubov method of averaging, with separation of motions (by a change of variables [1–3]) over relatively longer time intervals. We will consider some approaches to the investigation of oscillatory systems in the standard form (in Bogolyubov's sense) and of systems containing fast and slow phases.

1. STATEMENT OF THE PROBLEM

Consider a non-linear oscillatory system in the Bogolyubov standard form [1, 2]

$$\dot{z} = \varepsilon Z(t, z), \qquad t \geqslant 0, \qquad z(0) = z^0, \qquad z \in D_z, \qquad 0 \leqslant \varepsilon \leqslant \varepsilon_0 \ll 1 \tag{1.1}$$

The vector valued function Z, of arbitrary dimension n_z, is assumed to be sufficiently smooth with respect to $z \in D_z$, where D_z is some connected set, and piecewise-continuous and 2π-periodic in t (t is the time or oscillation phase). The averaged system of the first approximation is constructed in the standard way [1, 2] and it is assumed that the averaged equations possess the structural properties defined below

$$\begin{gathered}\dot{\xi} = \varepsilon X_0(\xi, \eta), \qquad \xi(0) = x^0; \qquad \dot{\eta} = \varepsilon Y_0(\xi, \eta), \qquad \eta(0) = y^0 \\ z = (x^T, y^T)^T, \qquad Z(t, z) \equiv (X^T(t, x, y), Y^T(t, x, y))^T \\ x, \xi \in D_x, \qquad y, \eta \in D_y, \qquad D_z = D_x \times D_y \\ X_0 = \langle X(t, \xi, \eta) \rangle \equiv 0; \qquad Y_0 = \langle Y(t, \xi, \eta) \rangle \not\equiv 0\end{gathered} \tag{1.2}$$

where x and y are vectors of arbitrary dimensions n_x and n_y, respectively ($n_z = n_x + n_y$), $n_y \geqslant 1$ and ξ and η are the "averaged" values of these vectors. The special case $n_y = 0$, i.e. $x = z$, needs separate discussion [4], see below. It follows from (1.2) [1, 2] that in the first approximation with respect to ε the average rate of change of the vector x equals zero, that is, $|x - x^0| = O(\varepsilon)$ for $t \sim 1/\varepsilon$, while the variable y, generally speaking, varies significantly: $|y - y^0| = O(1)$. It is more convenient to treat the equation for η in the slow time $\tau = \varepsilon t$.

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For applications, however, one is frequently interested in the behaviour of system (1.1) over an interval which is significantly longer with respect to negative degrees of ε, say $t \in I_2$ ($I_k = [0, L/\varepsilon^k]$, $k = 0, 1, 2, \ldots$), over which the variable x may vary significantly. In the general case it is difficult to solve this problem by the method of averaging. We will therefore consider the situation, encountered in applied problems, in which the equation for η when $\xi = \text{const}$ ($\xi \in D_x$) admits of a complete family of single-frequency periodic solutions in the slow time τ [1]

$$\eta = \eta_0(\varphi, \xi, \zeta), \qquad \varphi = \omega(\xi, \zeta)\tau + \varphi^0, \qquad \tau = \varepsilon t, \qquad \eta \in D_y \tag{1.3}$$

where ζ and φ^0 are integration constants and φ is the scalar phase; the dimension of ζ, $\zeta \in D_\zeta$, equals $n_y - 1$. The function η_0 describes oscillatory or rotatory-oscillatory motions and is assumed to be 2π-periodic in the usual sense [1–3] with respect to φ. The phase φ is a slow variable with respect to t: when $t \sim 1/\varepsilon^2$ it receives an increment $\delta\varphi \sim 1/\varepsilon$. Thus, on the assumption that conditions (1.2) and (1.3) are satisfied, one can formulate the problem of constructing and justifying an averaging scheme of higher degree in ε, as compared with the standard approach [1–3]. Such a modification of the method of averaging was first used in [5] to investigate non-resonance rotations of a triaxial artificial satellite in an elliptic orbit.

2. REDUCTION TO A SYSTEM WITH FAST AND SLOW PHASES

Following the procedure for a change of variables in [1–3], we apply a transformation $(x, y) \to (\xi, \eta)$ close to the identity transformation

$$\begin{gathered} x = \xi + \varepsilon U(t, \xi, \eta), \qquad y = \eta + \varepsilon V(t, \xi, \eta) \\ U = \int_0^t X(s, \xi, \eta)ds, \qquad V = \int_0^t [Y(s, \xi, \eta) - Y_0(\xi, \eta)]ds \end{gathered} \tag{2.1}$$

The functions X and Y ($X_0 \equiv 0$, $Y_0 \not\equiv 0$) are defined as in (1.2).

Differentiating expressions (2.1) along trajectories of Eqs (1.1), we obtain the Cauchy problem

$$\begin{gathered} \dot{\xi} = \varepsilon^2 \Xi(t, \xi, \eta, \varepsilon), \qquad \xi(0) = x^0 \\ \dot{\eta} = \varepsilon Y_0(\xi, \eta) + \varepsilon^2 \mathrm{H}(t, \xi, \eta, \varepsilon), \qquad \eta(0) = y^0 \\ \Xi = [I + \varepsilon U'_\xi - \varepsilon^2 U'_\eta (I + \varepsilon V'_\eta)^{-1} V'_\xi]^{-1} [\Delta X - U'_\eta (I + \varepsilon V'_\eta)^{-1} (Y_0 + \Delta Y)] \\ \mathrm{H} = (I + \varepsilon V'_\eta)^{-1} (\Delta Y - V'_\eta Y_0 - \varepsilon V'_\xi \Xi) \\ \varepsilon \Delta X = X - (X), \qquad \varepsilon \Delta Y = Y - (Y) \end{gathered} \tag{2.2}$$

The expressions (X) and (Y) mean that the functions X and Y are evaluated at $x = \xi$, $y = \eta$.

It follows from the relations (2.1) and (2.2) that the right-hand sides of the equations are 2π-periodic with respect to t and sufficiently smooth with respect to $\xi \in D_x$ and $\eta \in D_y$, $0 < \varepsilon \leqslant \varepsilon_0$. Up to terms $O(\varepsilon^2)$, they are identical with Eqs (1.2).

We transfer from variables ξ, η to variables ξ, ζ, φ by means of a non-singular substitution

$$\xi = \xi, \qquad \eta = \eta_0(\varphi, \xi, \zeta) \qquad \left(\frac{\partial \eta_0}{\partial \varphi} \omega(\xi, \zeta) \equiv Y_0(\xi, \eta_0) \right) \tag{2.3}$$

where η_0 and ω are known functions according to (1.3). Let us combine the vectors ξ and ζ into a single $(n_x + n_y - 1)$-dimensional vector $\alpha = (\xi^T, \zeta^T)^T$; the standard procedure yields the system with slow scalar phase φ

$$\begin{gathered} \dot{\alpha} = \varepsilon^2 \mathrm{A}(t, \alpha, \varphi, \varepsilon), \qquad \alpha(0) = \alpha^0, \qquad \alpha, \alpha^0 \in D_\alpha \\ \dot{\varphi} = \varepsilon\omega(\alpha) + \varepsilon^2 \Phi(t, \alpha, \varphi, \varepsilon), \qquad \varphi(0) = \varphi^0 \pmod{2\pi}, \qquad |\varphi| < \infty \\ \mathrm{A} = (\Xi^T, \mathrm{Z}^T)^T, \qquad (\Phi, \mathrm{Z}^T)^T = \left\| \frac{\partial \eta_0}{\partial \varphi}, \frac{\partial \eta_0}{\partial \zeta} \right\|^{-1} \left(\mathrm{H} - \frac{\partial \eta_0}{\partial \xi} \Xi \right) \end{gathered} \tag{2.4}$$

The component ξ of the vector α is described by Eq. (2.2), into whose right-hand side, that is, the function Ξ, the known expression (2.3) for η_0 has been substituted. For the component ζ, the right-hand side of the equation is defined by the function Z; to find it one should invert the matrix $(\partial\eta_0/\partial(\varphi, \zeta))$ according to (2.4), which also defines the scalar function Φ. The initial values $\zeta(0) = \zeta^0$ and $\varphi(0) = \varphi^0$ are found from relation (2.3) for η: $y^0 = \eta_0(\varphi^0, x^0, \zeta^0)$, where φ^0 is defined apart from 2π. The right-hand sides of system (2.4) are smooth functions of α, φ and ε and 2π-periodic functions of t and φ.

To investigate system (2.4) for $t \in I_2$, we apply the scheme of separation of slow and fast motions. The separation of variables may be incomplete (partial), in which case the system of equations for the variables corresponding to α and φ is coupled and the variable t is separated. For further complete separation of the variables α and φ one can use the averaged system not containing the argument t. As in the classical method of averaging for $t \in I_1$, the corresponding asymptotic expansions should not contain singular terms in the extended interval $t \in I_2$.

3. SEPARATION OF THE FAST PHASE

Let us regard α and φ as slow variables and the argument t as a fast phase. We transform the variables α and φ to β and ψ, where β and ψ are quantities averaged over t, governed by equations that do not contain t

$$\begin{gathered}
\alpha = \beta + \varepsilon^2\Pi(t,\beta,\psi,\varepsilon) = \beta + \varepsilon^2\Pi_2(t,\beta,\psi) + \varepsilon^3\Pi_3 + \ldots + \varepsilon^k\Pi_k + \varepsilon^{k+1}\ldots \\
\varphi = \psi + \varepsilon^2\Gamma(t,\beta,\psi,\varepsilon) = \psi + \varepsilon^2\Gamma_2(t,\beta,\psi) + \varepsilon^3\Gamma_3 + \ldots + \varepsilon^{k-1}\Gamma_{k-1} + \varepsilon^k\ldots \\
\dot\beta = \varepsilon^2 \mathrm{B}(\beta,\psi,\varepsilon) = \varepsilon^2[\mathrm{B}_0(\beta,\psi) + \varepsilon \mathrm{B}_1 + \ldots + \varepsilon^{k-2}\mathrm{B}_{k-2}] + \varepsilon^{k+1}\ldots \\
\dot\psi = \varepsilon\omega(\beta) + \varepsilon^2\Psi(\beta,\psi,\varepsilon) = \varepsilon\omega(\beta) + \varepsilon^2[\Psi_0(\beta,\psi) + \varepsilon\Psi_1 + \ldots + \varepsilon^{k-3}\Psi_{k-3}] + \varepsilon^k\ldots \\
k \geqslant 2, \quad \Gamma_j, \Psi_j \equiv 0, \quad j \leqslant 1, \quad \omega \not\equiv \text{const}
\end{gathered} \tag{3.1}$$

The unknown coefficients Π_j and Γ_j of asymptotic expansions (3.1) and the corresponding right-hand sides of the equations for β and ψ are obtained in the standard way [1, 2] by differentiating α and φ with respect to t, substituting into (2.4) and equating the coefficients of like powers of ε^j

$$\begin{gathered}
(I + \varepsilon^2\Pi'_\beta)\mathrm{B} + \Pi'_\psi(\varepsilon\omega + \varepsilon^2\Psi) = \mathrm{A}(t, \beta + \varepsilon^2\Pi, \psi + \varepsilon^2\Gamma, \varepsilon) - \Pi'_t \\
\varepsilon^3\Gamma'_\beta\mathrm{B} + (1 + \varepsilon^2\Gamma'_\psi)(\omega(\beta) + \varepsilon\Psi) = \omega(\beta + \varepsilon^2\Pi) + \varepsilon\Phi(t, \beta + \varepsilon^2\Pi, \psi + \varepsilon^2\Gamma, \varepsilon) - \varepsilon\Gamma'_t \\
\mathrm{B} = \mathrm{B}_0 + \varepsilon\mathrm{B}_1 + \varepsilon^2\mathrm{B}_2 + \ldots, \qquad \Psi = \Psi_0 + \varepsilon\Psi_1 + \ldots \\
\Pi = \Pi_2 + \varepsilon\Pi_3 + \varepsilon^2\Pi_4 + \ldots, \qquad \Gamma = \Gamma_2 + \varepsilon\Gamma_3 + \ldots
\end{gathered} \tag{3.2}$$

The unknown functions B, Ψ, Π and Γ may be determined recurrently from (3.2) to any degree of accuracy in ε, as determined by the smoothness of the right-hand sides of system (2.4). In particular, one can use the procedure of expansion in powers of the parameter ε. In this case Eqs (3.2) split at each stage and the required expressions have the form

$$\begin{gathered}
\mathrm{B}_0(\beta,\psi) = \mathrm{A}_0(\beta,\psi) = \langle \mathrm{A}(t,\beta,\psi,0)\rangle, \qquad \Psi_0(\beta,\varphi) = \Phi_0(\beta,\psi) = \langle\Phi(t,\beta,\psi,0)\rangle \\
\Pi_2(t,\beta,\psi) = \int_0^t ((\mathrm{A}) - \mathrm{A}_0)ds, \qquad \Gamma_2(t,\beta,\psi) = \int_0^t ((\Phi) - \Phi_0)ds \\
\mathrm{B}_1 = \langle(\mathrm{A}'_\varepsilon)\rangle - (\omega)\langle\Pi'_{2\varphi}\rangle, \qquad \Psi_1 = \langle(\Phi'_\varepsilon)\rangle + (\omega')\langle\Pi_2\rangle - (\omega)\langle\Gamma'_{2\psi}\rangle \\
\Pi_3 = \int_0^t ((\mathrm{A}'_\varepsilon) - (\omega)\Pi'_{2\varphi} - \mathrm{B}_1)ds \\
\mathrm{B}_2 = \frac{1}{2}\langle \mathrm{A}''_{\varepsilon^2}\rangle + \langle(\mathrm{A}'_\alpha)\Pi_2\rangle + \langle(\mathrm{A}'_\varphi)\Gamma_2\rangle - \langle\Pi'_{2\beta}\rangle\mathrm{B}_0 - (\omega)\langle\Pi'_{3\psi}\rangle - \langle\Pi'_{2\varphi}\rangle\Psi_0
\end{gathered} \tag{3.3}$$

The following coefficients B_j, Π_j, Ψ_j and Γ_j are calculated similarly. They enable us to construct the system of equations in any approximation in terms of ε for $t \in I_2$ for the variables β and ψ, the slow phase ψ being determined with lower accuracy (with error ε^{j-1}) compared with the variable β. This is the case also for the initial variables α and φ. Similarly, as in the classical scheme [1, 2] for constructing the jth approximation, the functions Π_{j+1} and Γ_j will not be needed. Thus, in the first approximation, that is, with error $O(\varepsilon)$ for α and $O(1)$ for φ, we have according to (3.3)

$$\dot{\beta} = \varepsilon^2 A_0(\beta, \psi), \quad \beta(0) = \alpha^0, \quad \dot{\psi} = \varepsilon\omega(\beta), \quad \psi(0) = \varphi^0, \quad t \in I_2 \tag{3.4}$$

The construction of the solution is reduced to the integration of the Cauchy problem (3.4); when doing so, one can introduce a slow argument $\tau = \varepsilon t$, where $\tau \in I_1$. The right-hand side of the autonomous system (3.4) is 2π-periodic with respect to the phase ψ, $\psi \sim 1/\varepsilon$. It is required to integrate these equations for $\tau \in I_1$ to within $O(\varepsilon)$ with respect to the slow variable β and $O(1)$ for the phase ψ. The system may be investigated by qualitative methods on a cylindrical surface. If the function $\omega(\beta)$ is bounded away from zero in the domain under consideration of variation of β, say $\omega \geqslant \omega_0 > 0$, then the desired approximate solution for $\beta(\varepsilon^2 t)$ can be found by introducing an argument ψ, $\psi \sim 1/\varepsilon$, and averaging with respect to ψ by the standard procedure of the method of averaging. The phase $\psi(\varepsilon t)$ is then determined from an implicit relationship by a simple quadrature [1–3].

The following proposition holds.

Theorem 1. A solution (exact or approximate) of Cauchy problem (3.4) in the range of variation of the slow argument $\tau \in I_1$ determines a solution of problem (2.4) to within $O(\varepsilon)$ in the slow variable α over the interval of the initial argument $t \in I_2$; the variable φ is determined to within $O(1)$. The slow variable x of the initial system (1.1) is calculated to within $O(\varepsilon)$ and the relatively fast variable y to within $O(1)$ over the indicated range of t.

The proof follows from the substitutions (2.1), (2.3) and (3.1) and estimates based on Gronwall's Lemma. The function A in (2.4) is required to satisfy a uniform Lipschitz condition with respect to α and to be continuous with respect to φ; the function Φ is assumed to be continuous in α and φ; the frequency ω must satisfy a uniform Lipschitz condition ($\alpha \in D_\alpha$, $0 \leqslant \varphi \leqslant 2\pi$). The proof is simplified if one requires A and ω to be differentiable with respect to $\alpha \in D_\alpha$.

Similarly, in the second approximation, to within $O(\varepsilon^2)$ with respect to α and x and $O(\varepsilon)$ with respect to φ and y for $t \in I_2$, one has the following Cauchy problem (see (3.1)–(3.3))

$$\begin{aligned} &\beta^\bullet = \varepsilon A_0(\beta, \psi) + \varepsilon^2 B_1(\beta, \psi), \quad \beta(0) = \alpha^0, \quad \tau \in I_1 \\ &\psi^\bullet = \omega(\beta) + \varepsilon\Psi_0(\beta, \psi), \quad \psi(0) = \varphi^0, \quad \tau = \varepsilon t \end{aligned} \tag{3.5}$$

The slow argument τ has been introduced into system (3.5) in order to reduce the system with rotating phase to standard form. The "perturbing functions" B_1 and Ψ_0 are defined in (3.3); this requires a higher degree of smoothness of the functions A, ω and Φ, that is, of the initial functions X and Y. It is required to construct a solution of the simplified system (3.5) for $\tau \in I_1$ to within $O(\varepsilon^2)$ with respect to β and $O(\varepsilon)$ with respect to ψ. To that end, as before, either numerical or analytical methods may be used. Once a first-approximation solution has been constructed, a second approximation is constructed by perturbation methods on the basis of the variational system [2, 3].

A similar procedure yields solutions, averaged with respect to t, in any, say the jth, approximation with respect to powers of ε in the range $t \in I_2$. According to (3.1) and (3.2), we have a jth approximation system in which terms $O(\varepsilon^{j+2})$ are dropped in the equation for the slow vector β and terms $O(\varepsilon^{j+1})$ for the slow phase ψ. Denoting differentiation with respect to the slow argument $\tau = \varepsilon t$ by a dot on the side, we obtain a standard system in Bogolyubov's sense [1–3]

$$\begin{aligned} &\beta^\bullet = \varepsilon A_0 + \varepsilon^2 B_1 + \ldots + \varepsilon^j B_{j-1}, \quad \beta(0) = \alpha^0 \\ &\psi^\bullet = \omega(\beta) + \varepsilon\Phi_0 + \varepsilon^2\Psi_1 + \ldots + \varepsilon^{j-1}\Psi_{j-1}, \quad \psi(0) = \varphi^0 \end{aligned} \tag{3.6}$$

We recall that the omitted terms of expansion (3.6) are also 2π-periodic functions of the fast argument $t = \tau/\varepsilon$.

Let us assume now that the solution of Cauchy problem (3.6) is constructed to within $O(\varepsilon^j)$ with respect to β and $O(\varepsilon^{j-1})$ with respect to ψ. The following proposition holds.

Theorem 2. According to (3.1)–(3.3), the functions $\beta(\tau, \varepsilon)$ and $\psi(\tau, \varepsilon)$ determine a solution of Cauchy problem (2.4), $\alpha(t, \varepsilon)$ and $\varphi(t, \varepsilon)$, to within $O(\varepsilon^j)$and $O(\varepsilon^{j-1})$, respectively, in the range $t \in I_2$. According to (2.1)–(2.3), the errors $O(\varepsilon^j)$ and $O(\varepsilon^{j-1})$ are valid for the variables x and y of the initial system (1.1), (1.2).

The proof is carried out by standard methods using integral inequalities (Gronwall's Lemma [1, 2]). To justify the approximation, the solution of the problem in the first approximation $x_{(1)}(t, x^0, y^0, \varepsilon)$, $y_{(1)}(t, x^0, y^0, \varepsilon)$ is required to belong to a bounded domain $D_x \times D_y$, together with some neighbourhood, over the relevant range of the argument, for sufficiently small values of $\varepsilon > 0$ and, in addition, it is necessary that the $(j - 1)$th derivatives of the functions X and Y (1.2) with respect to x and y satisfy Lipschitz conditions.

The fast phase (argument) t can also be separated by successive approximations. After reducing relations (3.2) to the form

$$
\begin{aligned}
\Pi_t' &= \mathrm{A}(t,\beta,\psi,0) - \mathrm{B} + \varepsilon F(t,\beta,\psi,\Pi,[\Pi],\Gamma,\mathrm{B},\Psi,\varepsilon) \\
\Gamma_t' &= \Psi(t,\beta,\psi,0) - \Psi + \varepsilon G(t,\beta,\psi,\Pi,\Gamma,[\Gamma],\mathrm{B},\Psi,\varepsilon)
\end{aligned}
\tag{3.7}
$$

where the expressions $[\cdot]$ denote linear differential operators with respect to β and ψ, the required functions B and Ψ in the equations for β and ψ and the functions Π and Γ defining the appropriate change of variables α and φ to β and ψ (3.1) are constructed by recurrence. At the first step of the procedure (for $\varepsilon = 0$) we have (see (3.2) and (3.3))

$$
\begin{aligned}
&\mathrm{B}_0(\beta,\psi) = \langle \mathrm{A}(t,\beta,\psi,0)\rangle \equiv \langle \mathrm{A}_{(0)}\rangle \\
&\Psi_{(0)}(\beta,\psi) = \langle \Phi(t,\beta,\psi,0)\rangle \equiv \langle \Psi_{(0)}\rangle \\
&\Pi_0(t,\beta,\psi) = \int_0^t (\mathrm{A}(s,\beta,\psi,0) - \langle \mathrm{A}_{(0)}\rangle)ds \\
&\Gamma_{(0)}(t,\beta,\psi) = \int_0^t (\Phi(s,\beta,\psi,0) - \langle \Phi_{(0)}\rangle)ds
\end{aligned}
\tag{3.8}
$$

The next approximations are described by similar relations

$$
\begin{aligned}
&\mathrm{B}_{j+1}(\beta,\psi,\varepsilon) = \langle \mathrm{A}(t,\beta,\psi,0) + \varepsilon F(t,\beta,\psi,\Pi_{(j)},[\Pi_{(j)}],\Gamma_{(j)}\mathrm{B}_{(j)},\Psi_{(j)},\varepsilon)\rangle \\
&\Psi_{j+1}(\beta,\psi,\varepsilon) = \langle \Psi(t,\beta,\psi,0) + \varepsilon G(t,\beta,\psi,\Pi_{(j)},\Gamma_{(j)},[\Gamma_{(j)}],\mathrm{B}_{(j)},\Psi_{(j)},\varepsilon)\rangle \\
&\Pi_{j+1}(t,\beta,\psi,\varepsilon) = \int_0^t (\mathrm{A}_{(0)} + \varepsilon F - \langle \mathrm{A}_{(0)} + \varepsilon F\rangle)ds \\
&\Gamma_{j+1}(t,\beta,\psi,\varepsilon) = \int_0^t (\Psi_{(0)} + \varepsilon G - \langle \Psi_{(0)} + \varepsilon G\rangle)ds
\end{aligned}
\tag{3.9}
$$

Expansions of formulae (3.9) yield functions which are the coefficients of formulae (3.2). Contrary to a well-known assertion [6, pp. 196, 197], we observe that the construction of the standard scheme of successive approximations involves imposing more stringent smoothness conditions on the functions A, ω and Ψ with respect to the variables α and φ as the order of the approximation increases. As is readily seen from (3.2), (3.7) and (3.9), this is because of the increase in the order of the derivatives with respect to the slow variables β and ψ. At the same time, the integration with respect to t, by (3.9), does not increase the degree of smoothness with respect to the independent variables β and ψ. Using successive approximations, we can construct the trajectories of the appropriate perturbed Cauchy problem with non-analytic right-hand sides [3]. In the general case, exact separation of variables does not occur even for analytic systems, because of "resonance between the slow and fast variables" [7, 8].

4. REMARKS AND POSSIBLE GENERALIZATION OF THE HIGHER-ORDER AVERAGING SCHEME

1. Let us consider a similar scheme in the case when the coefficients B_i satisfy the following conditions

$$
X_0 = \mathrm{B}_0 = \mathrm{B}_1 = \ldots = \mathrm{B}_{k-3} \equiv 0, \qquad \mathrm{B}_{k-2} \not\equiv 0, \qquad k \geqslant 3
\tag{4.1}
$$

when constructing the averaged system (3.1). This means that the term of order ε^k in the equation for the variable β is non-zero; by virtue of conditions (4.1), we obtain a system of the form

$$\begin{aligned} \dot\beta &= \varepsilon^k B_{k-2}(\beta,\psi)+\varepsilon^{k+1}B_k^*(t,\beta,\psi,\varepsilon) \\ \dot\psi &= \varepsilon\omega(\beta)+\varepsilon^2(\Psi_0+\varepsilon\Psi_1+\ldots+\varepsilon^{k-3}\Psi_{k-3})+\varepsilon^k\Psi_k^*(t,\beta,\psi,\varepsilon) \end{aligned} \tag{4.2}$$

Then, dropping terms $O(\varepsilon^{k+1})$ in the equation for β in system (4.2) and terms $O(\varepsilon^k)$ in the equation for ψ, we obtain the first-approximation system. Its solution $\beta_1(t, \alpha^0, \varphi^0, \varepsilon)$, $\psi_1(t, \alpha^0, \varphi^0, \varepsilon)$, to within $O(\varepsilon)$ with respect to α and $O(1)$ with respect to φ, determines a solution of system (2.4) in the range $t \in I_k$. In particular, if $\omega(\beta) \neq 0$, averaging with respect to φ reduces this system to the form

$$\begin{aligned} \dot\beta &= \varepsilon^k\langle B_{k-2}\rangle \\ \dot\psi &= \varepsilon\omega(\beta)+\varepsilon^2\langle\Psi_0+\varepsilon\Psi_1+\ldots+\varepsilon^{k-3}\Psi_{k-1}\rangle_{(k-1)} \equiv \varepsilon\omega_{(k)}(\beta,\varepsilon) \\ \beta &= \beta_1(\varepsilon^k t,\alpha^0)+O(\varepsilon), \qquad \psi = \varphi^0+\varepsilon\int_0^t \omega_{(k)}(\beta_1,\varepsilon)dt_1+O(1) \end{aligned} \tag{4.3}$$

The fact that the solution of system (4.3) is close to that of system (4.2) is established by the standard method of averaging with respect to the "fast phase" ψ, after dividing $\dot\beta$ by $\dot\psi$.

2. Similar expansions for the slow variable β and slow phase φ may be carried out in the case when $\omega(\alpha) \equiv 0$ as well, and the averaged system of type (4.2) has the form

$$\begin{aligned} \dot\beta &= \varepsilon^k B_{k-2}(\beta,\psi)+\varepsilon^{k+1}B_k^*(t,\beta,\psi,\varepsilon) \\ \dot\psi &= \varepsilon^{l+2}\Psi_l(\beta,\psi)+\ldots+\varepsilon^{k-1}\Psi_{k-3}(\beta,\psi)+\varepsilon^k\Psi_k^*(t,\beta,\psi,\varepsilon) \end{aligned} \tag{4.4}$$

where $0 \leqslant l \leqslant k-2$, $k \geqslant 3$; in particular, when $k = 3$, $l = 0$, we obtain the first-approximation system

$$\dot\beta = \varepsilon^3 B_1(\beta,\psi), \qquad \dot\psi = \varepsilon^2\Psi_0(\beta,\psi) \tag{4.5}$$

which must be integrated over the interval $\tau \in I_1$ after the slow argument $\tau = \varepsilon^2 t$ has been introduced. Assuming that $|\Psi_0| \geqslant c > 0$, we obtain the following equation for the vector β from Eqs (4.5)

$$\frac{d\beta}{d\psi} = \frac{\varepsilon B_1(\beta,\psi)}{\Psi_0(\beta,\psi)}, \qquad 0 \leqslant \psi-\varphi^0 \leqslant \frac{\Theta}{\varepsilon}$$

which may be simplified and investigated approximately by the standard procedure of the method of averaging (with respect to ψ). Using the equation averaged with respect to ψ, one constructs a first-approximation solution $\beta_1(\varepsilon(\psi-\varphi^0), \alpha^0)$ for β, and then uses the relation

$$\varepsilon^2 t = \tau = \int_{\varphi^0}^{\psi} \frac{d\gamma}{\Psi_0(\beta_1(\varepsilon\gamma),\gamma)}$$

to determine the function $\psi = \psi_1(\tau, \alpha^0, \varphi^0)$ to within $O(1)$ in the interval $\tau \in I_1$. After substituting ψ_1 into the expression for β_1 one finds the slow vector to within $O(\varepsilon)$.

Similar constructions may be carried out in the general case of system (4.4) in the range $t \in I_k$.

3. Together with the version considered above, in which there is one slow phase (see (1.3) and (2.4)), one can investigate a more complicated system, containing a hierarchy of slow phases of the form

$$\begin{aligned} \dot\alpha &= \varepsilon^k A(t,\alpha,\varphi,\varepsilon), \qquad \dim\alpha = n, \qquad \dim\varphi = m \geqslant 0 \\ \dot\varphi_1 &= \varepsilon\omega_1(\alpha,\varphi_2,\ldots,\varphi_m)+\varepsilon^2\Phi_1(t,\alpha,\varphi,\varepsilon) \\ \dot\varphi_2 &= \varepsilon^2\omega_2(\alpha,\varphi_3,\ldots,\varphi_m)+\varepsilon^3\Phi_2(t,\alpha,\varphi,\varepsilon) \\ &\ldots \\ \dot\varphi_m &= \varepsilon^m\omega_m(\alpha)+\varepsilon^{m+1}\Phi_m(t,\alpha,\varphi,\varepsilon), \qquad m \leqslant k-1 \end{aligned} \tag{4.6}$$

The functions A, Φ_j and ω_j are naturally assumed to be 2π-periodic with respect to the phases, and the constants ω_j are non-zero. In the general case, the procedure for separation of variables in system (4.6) is extremely complicated. In any specified case, however, one can construct a suitable substitution which enables the integration of the slower variables to be separated from that of the faster ones. If $m = 0$, one assumes that the system involves only one fast variable t, with respect to which the equations are averaged; it varies in the range $t \in I_k$. Note that in the general case one can also apply the asymptotic expansion procedure of the multiple-scales method [9].

4. Instead of system (1.1) one can take a system of more general form [1, 2, 8]

$$\dot{x} = X(x, \varepsilon), \quad x(0) = x^0, \quad \dim x = n \geqslant 2 \tag{4.7}$$

which, taken with $\varepsilon = 0$, admits of a non-degenerate family of periodic solutions $x_0(\theta, a)$, where the scalar phase is $\theta = \nu(a)t + \theta^0$, $\nu \geqslant \nu_0 > 0$, and a is an m-vector of arbitrary constants from some domain, $a \in D$, $1 \leqslant m \leqslant n-1$. If $m < n-1$, it is assumed that this family is asymptotically (exponentially) stable [1, 2]. Then the traditional substitution exists [1, 10, 11]

$$x = x_0(\theta, a) + \frac{1}{2}[N(\theta, a)h + N^*(\theta, a)h^*], \; N'_\theta \nu + NH = (X')N, \; N^{*\prime}_\theta \nu + N^* H^* = (X'_x)N^* \tag{4.8}$$

where N is a complex matrix which is a periodic function of θ, h is a vector and the asterisk denotes complex conjugation, and this substitution transforms system (4.7) into a system of equations in a certain neighbourhood of the local integral manifold

$$\begin{aligned} \dot{a} &= A(a, \theta, h, \varepsilon), \quad a(0) = a^0, \quad A = O(|\varepsilon| + |h|^2) \\ \dot{\theta} &= \nu(a) + \Theta(a, \theta, h, \varepsilon), \quad \theta(0) = \theta^0, \quad \Theta = O(|\varepsilon| + |h|^2) \\ \dot{h} &= B(a)h + H(a, \theta, h, \varepsilon), \quad h(0) = h^0, \quad H = O(|\varepsilon| + |h|^2) \end{aligned} \tag{4.9}$$

Under the substitution (4.8), the unknown variable x remains real; the functions A, Θ and H in system (4.9) are also real. In addition, the characteristic exponents of the matrix $B(a)$ have negative real parts. For sufficiently small values of $|\varepsilon|$ and $|h^0|$, the solutions tend to a stable integral manifold; in a relatively short time, of the order of $\ln |\varepsilon|^{-1}$, they approach as close as desired to the manifold described by the equation

$$h = \varepsilon h_1(\alpha, \theta) + \varepsilon^2 h_2(a, \theta) + \varepsilon^3 \ldots \tag{4.10}$$

where the functions h_j are constructed in the standard way [1, 10, 11].

The accuracy of the construction of $h(a, \theta, \varepsilon)$ is determined by the degree of smoothness of system (4.9), that is, of the initial system (4.7). After substituting the functions (4.10) into the equations for a and θ and dividing $\dot{a}$ by $\dot{\theta}$, one obtains a system of the standard form (1.1), to within the required accuracy with respect to powers of ε ($t = \theta$ is the argument). Under suitable assumptions, this system may be investigated by means of the standard procedure of the method of averaging or higher-order averaging schemes, see above. In many cases encountered in applications, the averaged equations for the slow variable a may be obtained using integrals of the unperturbed system (4.7) [2, 3].

5. In the special case of system (2.4) with $\omega(\alpha) = \nu = \text{const}$, the variables α and φ may be determined to within the same degree of accuracy in ε. In this case, one has to construct the coefficients Π_{k-1}, B_{k-2}, Γ_{k-1}, Ψ_{k-2} in expansions (3.2), which leads to the same error $O(\varepsilon^k)$ in determining α and φ in the range $t \in I_k$; at the same time, $|\alpha - \alpha^0| \sim 1$, $|\varphi - \varphi^0| \sim 1/\varepsilon$.

6. The procedure for separating the fast phase t (averaging with respect to t) is carried out by analogy with the case of a system of the form of (2.4) in which φ is a vector phase of arbitrary dimensionality $n_\varphi \leqslant n_y$. The transfer from system (2.2) to equations with a slow phase φ by a change of variables of type (2.3) is accomplished by similar means. However, the analysis of Eqs (2.4), averaged with respect to t, using the procedure (3.1), e.g. systems (3.4)–(3.6), in the case of a vector phase φ in the range $t \in I_2$ leads to well-known difficulties ("small denominators") typical of essentially non-linear multifrequency systems [1, 2, 8].

7. When analysing quasi-linear oscillatory systems, one can investigate a resonance situation in which one of the frequencies has much larger frequency mismatch than the others. Under suitable assumptions regarding the properties of the perturbing effects, averaged over t (see Section 2), the multifrequency

system may be reduced to the form of (2.4), where $\varepsilon\omega$ = const is the frequency mismatch. This system may be simplified and studied for $t \in I_2$ using the approach outlined in Section 3; see examples in Section 5.

5. EXAMPLES

We will present the solution in the first approximation for oscillatory systems of type (1.1) that can be represented in the form of (2.4) and are readily investigated using second-order averaging schemes.

Model example. First, for illustrative purposes, we consider a two-dimensional system of the form

$$\begin{aligned} \dot{x} &= \varepsilon f(x,y)\sin(t+\theta(x,y)), \qquad x(0)=x^0 \\ \dot{y} &= \varepsilon\gamma(x,y), \qquad y(0)=y^0 \end{aligned} \tag{5.1}$$

where f, θ and γ are smooth functions of x and y, 2π-periodic with respect to the slow phase $\varphi = y$. Applying transformation (2.3), we reduce system (5.1) to the form of (2.4)

$$\begin{aligned} \dot{a} &= \varepsilon^2(f_a'u\sin(t+\theta)+fu\theta_a'\cos(t+\theta)-f_\varphi'\gamma(\cos\theta-\cos(t+\theta))- \\ &-f\theta_\varphi'\gamma(\sin(t+\theta)-\sin\theta))+\varepsilon^3\ldots \\ \dot{\varphi} &= \varepsilon\gamma(a,\varphi)+\varepsilon^2\ldots, \qquad f=f(a,\varphi), \qquad \theta=\theta(a,\varphi), \qquad \varphi=y \\ x &= a+\varepsilon u, \qquad u=f(a,\varphi)(\cos\theta(a,\varphi)-\cos(t+\theta(a,\varphi))) \end{aligned} \tag{5.2}$$

Applying the averaging procedure of Section 3, we obtain equations that do not contain the argument t explicitly

$$\begin{aligned} \dot{\xi} &= -\varepsilon^2\left(\frac{1}{2}f^2\theta_\xi'+\gamma(f\cos\theta)_\varphi'\right), \qquad \xi(0)=x^0 \\ \dot{\varphi} &= \varepsilon\gamma(\xi,\varphi), \qquad \varphi(0)=\varphi^0 \end{aligned} \tag{5.3}$$

The variable φ is not transformed in the first approximation under consideration, and we preserve the old notation for this variable.

Equations (5.3) admit of the introduction of the argument $\tau = \varepsilon t$; in the interval $\tau \sim 1/\varepsilon$ the variable ξ, and together with it also a and x, receive an increment of the order of unity in the general case; the slow phase varies by an amount $O(\varepsilon^{-1})$. To investigate Eqs (5.3), one can apply analytical and qualitative phase plane methods, as well as numerical methods. If the function γ is non-zero, one obtains the standard equation for the derivative $d\xi/d\varphi$, and this equation can be dealt with using the classical averaging procedure with respect to φ in the interval $\varphi \sim 1/\varepsilon$. In the first approximation with respect to ε, we obtain the equation

$$\frac{d\xi}{d\varphi} = \varepsilon\Xi(\xi), \qquad \Xi \equiv -\frac{1}{2}\left\langle\frac{f^2\theta_\xi'}{\gamma}\right\rangle_\varphi, \qquad \xi(\varphi^0)=x^0, \qquad |\gamma| \geqslant \gamma_* > 0 \tag{5.4}$$

which admits of separation of variables and integration in quadrature. As a result one obtains expressions for the unknown solution to within $O(\varepsilon)$ in the range $t \in I_2$, of the following form

$$\int_{\xi^0}^{\xi}\frac{d\zeta}{\Xi(\zeta)} = \varepsilon(\varphi-\varphi^0), \qquad \xi=\xi_0(\varepsilon\varphi), \qquad \int_{\varphi^0}^{\varphi}\frac{d\psi}{\gamma(\xi_0(\varepsilon\psi),\psi)} = \varepsilon t \tag{5.5}$$

It is assumed in (5.4) and (5.5) that $\Xi \not\equiv 0$. These relations indicate that if $t \sim \varepsilon^{-2}$, the variables x and y evolve significantly. The standard procedure over the interval $t \sim 1/\varepsilon$ leads to simpler expressions: $x = x^0 + O(\varepsilon), \dot{y} = \varepsilon\gamma(x^0, y) + O(\varepsilon^2)$.

Forced quasi-linear oscillations. Let us consider perturbed periodic motions of a non-linear oscillator in dimensionless variables

$$\ddot{q}+Q(q)=P(t)-\Lambda\dot{q}, \qquad Q(0)=0 \tag{5.6}$$

taking place in a small neighbourhood of the point $q = \dot{q} = 0$, where q is a generalized coordinate and $\dot{q}$ is the velocity; the restoring force Q is assumed to be a fairly continuous function. The two-frequency action $P(t)$ is assumed to be small and a 2π-periodic function of t; $\Lambda > 0$ is a small dissipation coefficient, which is assumed to be constant. We introduce a small parameter $\varepsilon > 0$ characterizing these assumptions. We have

$$\begin{aligned}
&\ddot{y}+\nu^2 y=\varepsilon Y(t,y,\dot{y},\varepsilon); \qquad q=\varepsilon y, \qquad \nu^2=Q'(0)>0\\
&Y=h\sin 2t+\alpha y^2+\varepsilon(f\sin(t+\varkappa)+\beta y^3-\lambda\dot{y})+\varepsilon^2\ldots\\
&P=\varepsilon^2 h\sin 2t+\varepsilon^3 f\sin(t+\varkappa)\\
&\alpha=-\frac{1}{2}Q''(0), \qquad \beta=-\frac{1}{6}Q'''(0), \qquad \Lambda=\varepsilon^2\lambda
\end{aligned} \tag{5.7}$$

The real parameters $h, f, \alpha, \beta, \varkappa, \lambda$ and ν are assumed to be of the order of unity, with ν close to unity: $\nu = 1 + \varepsilon\gamma$, $\gamma - 1$. Changing to amplitude-phase variables in system (5.7), we have

$$\begin{aligned}
&\dot{a}=\varepsilon A(t,a,\psi,\varepsilon), \qquad A\equiv -F\sin\psi\\
&y=a\cos\psi, \qquad \dot{y}=-a\sin\psi, \qquad a>0\\
&\dot{\psi}=1+\varepsilon\Psi(t,a,\psi,\varepsilon), \qquad \Psi\equiv -Fa^{-1}\cos\psi\\
&F=Y-2\gamma y-\varepsilon\gamma^2 y
\end{aligned} \tag{5.8}$$

Instead of the phase ψ, we introduce the mismatch $\theta = \psi - t$ and represent system (5.8) in the form of Eqs (1.1), for which properties (1.2) are valid: $\langle(A)\rangle \equiv 0$, $\langle(\Psi)\rangle = \gamma$. An expression of the form $(\cdot)$ means that we are considering the function F with $\varepsilon = 0$. By virtue of the above relations for the averages $\langle(A)\rangle$ and $\langle(\Psi)\rangle$, a substitution of type (2.1) yields a system of the form (2.4)

$$\begin{aligned}
&\dot{\xi}=\varepsilon^2((A'_\xi)U_0+(A'_\delta)V_0+(A'_\varepsilon)-U'_{0\delta}\gamma)+\varepsilon^3\ldots\\
&\dot{\delta}=\varepsilon\gamma+\varepsilon^2\ldots, \qquad a=\xi+\varepsilon U_0, \qquad \theta=\delta+\varepsilon V_0\\
&U_0=-\left(\frac{2}{3}h(\sin^3 t\cos\delta+(1-\cos^3 t)\sin\delta)+\gamma\xi(\sin^2\delta-\sin^2(t+\delta))+\right.\\
&\left.+\frac{1}{3}\alpha\xi^2(\cos^3\delta-\cos^3(t+\delta))\right)\\
&V_0=-\left(\frac{2}{3}\frac{h}{\xi}((1-\cos^3 t)\cos\delta-\sin^3 t\sin\delta)-\right.\\
&\left.-\frac{1}{2}\gamma(\sin 2(t+\delta)-\sin 2\delta)+\alpha\xi(\sin(t+\delta)-\sin\delta+\frac{1}{3}\sin^3\delta-\sin^3(t+\delta))\right)
\end{aligned} \tag{5.9}$$

After averaging the equation for ξ in (5.9) with respect to t, one obtains a first-approximation system of type (3.4)

$$\begin{aligned}
&\dot{\xi}=\varepsilon^2\left(-\frac{1}{4}\gamma^2\xi-\frac{1}{2}f\cos(\varkappa-\delta)-\frac{1}{2}\lambda\xi+\frac{2}{3}h\gamma\cos\delta+\gamma^2\xi\sin 2\delta-\alpha\gamma\xi^2\cos^2\delta\sin\delta\right),\\
&\xi(0)=a^0\\
&\dot{\delta}=\varepsilon\gamma, \qquad \delta(0)=\varphi^0; \qquad t\in I_2
\end{aligned} \tag{5.10}$$

Further simplification of system (5.10) involves averaging the equation for ξ with respect to δ, according to the approach outlined in Section 3, for $\gamma \sim 1$. As a result one obtains a very simple equation, linear in ξ, integration of which yields the desired first-approximation solution

$$\xi = a^0 \exp\left(-\frac{1}{2}\varepsilon^2 t\left(\frac{\gamma^2}{2} + \lambda\right)\right), \qquad \delta = \varphi^0 + \varepsilon\gamma t \tag{5.11}$$

$$|a - \xi| \leq C\varepsilon, \quad |\theta - \delta| \leq C, \quad t \in I_2$$

It is interesting to observe that the frequency mismatch $\varepsilon\gamma$ leads to additional damping of the oscillations, irrespective of its sign. For the range $t \in I_1$ the standard averaging procedure gives $\xi = a^0 + O(\varepsilon)$.

Parametric oscillations of a pendulum. Consider the two-dimensional oscillations of a physical pendulum whose suspension point is being moved in a given way [3]. In standard notation and terms, we have the equation of motion

$$J\ddot{\alpha} + Mgl\sin\alpha = -Ml(\ddot{x}_0\cos\alpha + \ddot{y}_0\sin\alpha) - \Lambda\dot{\alpha} \tag{5.12}$$

Let us assume that the acceleration $(\ddot{x}_0, \ddot{y}_0)$ of the suspension point in the (x, y) plane varies periodically with frequency ν. Then, replacing the time argument t by the phase $\theta = \nu t$, we bring equation (5.12) to dimensionless form

$$\alpha'' + (N^2 + \varepsilon h''(\theta))\sin\alpha = -\varepsilon s''(\theta)\cos\alpha - \varepsilon^2\lambda\alpha' \tag{5.13}$$

$$N^2 = \frac{Mgl}{J\nu^2}, \quad \varepsilon h'' \equiv \frac{Ml\ddot{y}_0}{J\nu^2}, \quad \varepsilon s'' \equiv \frac{Ml\ddot{x}_0}{J\nu^2}, \quad \varepsilon^2\lambda = \frac{\Lambda}{J\nu}$$

where the primes denote differentiation with respect to θ and the small parameter ε characterizes the magnitude of the accelerations of the pendulum axis along the x and y axes and the smallness of the reduced dissipation coefficient Λ.

Equation (5.13) describes a large class of forced and parametric rotatory-vibratory motions of the pendulum, which have been investigated in a large number of publications. Considerable attention has been devoted to the case of fast vibrations of the suspension point ($N \sim \varepsilon$); below we will consider the problem of parametric small oscillations under the following assumptions (γ, $\varkappa$ = const)

$$\alpha = \varepsilon z, \quad h'' = -\cos 2\theta, \quad N = 2 + \varepsilon\gamma, \quad s'' = -4\varepsilon^2 d\sin(2\theta + \varkappa)$$

$$z'' + (4 - \varepsilon\cos 2\theta)z = -4\varepsilon\gamma z - \varepsilon^2\left(4\gamma^2 z + \lambda z' - \frac{2}{3}z^3 + 4d\sin(2\theta + \varkappa)\right) + \varepsilon^3 \ldots \tag{5.14}$$

According to relations (5.14), parametric oscillations are being investigated in an ε-neighbourhood of the second resonance zone. Standard methods convert system (5.14) to a system with rotating phase ψ, similar to system (5.8). We have

$$a' = \varepsilon A(\theta, a, \psi, \varepsilon), \quad z = a\cos\psi, \quad A = -\frac{1}{2}F\sin\psi$$

$$\psi' = 2 + \varepsilon\Psi(\theta, a, \psi, \varepsilon), \quad z' = -2a\sin\psi, \quad \Psi = -\frac{1}{2}Fa^{-1}\cos\psi \tag{5.15}$$

$$F \equiv z(\cos 2\theta - 4\gamma) - \varepsilon(4\gamma^2 z + \lambda z' - \frac{2}{3}z^3 + 4d\sin(2\theta + \varkappa))$$

The substitution $\psi = 2\theta + \varphi$ brings system (5.15) to the form of (1.1), for which conditions (1.2) are satisfied: $\langle(A)\rangle_\theta \equiv 0$, $\langle(\Psi)\rangle_\theta = \gamma$. The slow variable a and phase φ are subject to a transformation of type (3.1), which yields separation of the argument θ to within $O(\varepsilon)$ for $\theta \sim 1/\varepsilon^2$. We obtain the following relations, similar to (2.1)–(2.4)

$$
\begin{aligned}
&a = \xi + \varepsilon U(\theta, \xi, \delta), \qquad \varphi = \delta + \varepsilon V(\theta, \xi, \delta) \\
&U = \int_0^\theta (A) d\theta_1, \qquad V = \int_0^\theta ((\Psi) - \gamma) d\theta_1 \\
&\xi' = \varepsilon^2((A'_a)U + (A'_\psi)V + (A'_\varepsilon) - \gamma U'_\delta) + \varepsilon^3 \ldots \\
&\delta' = \varepsilon\gamma + \varepsilon^2 \ldots
\end{aligned}
\tag{5.16}
$$

Then, averaging the right-hand side of the equation for ξ with respect to the explicitly occurring argument θ, we obtain the aforementioned system (3.1), which is however very cumbersome. When $\gamma \sim 1$ ("far" from parametric resonance), after averaging with respect to the slow phase δ with the same relative error, we obtain the expressions

$$
a = \xi = a^0 \exp\left(-\frac{\varepsilon^2}{2}\lambda\theta\right) + O(\varepsilon), \qquad \delta = \varepsilon\gamma\theta + O(1), \qquad \theta \sim \frac{1}{\varepsilon^2}
\tag{5.17}
$$

It follows from these expressions that there will be no significant (i.e. of the order of $O(1)$) parametric and external effect on the oscillations of system (5.14) at $t \sim 1/\varepsilon^2$, provided the frequency mismatch is $\varepsilon\gamma$, $\gamma \sim 1$. The amplitude of the oscillations decays exponentially with respect to θ, with exponent $-{}^1\!/_2\varepsilon^2\lambda$. Of course, the situation changes radically if $\gamma \sim \varepsilon$, since under that condition the system will experience parametric and external resonances [4], due to vertical and horizontal vibrations, respectively, of the suspension point. Thus, investigating the oscillations of a pendulum with moving suspension point in the range $t \in I_2$, one observes mechanical effects due to significant evolution of the slow variables. These cannot be established by using the standard approach ($t \sim 1/\varepsilon$).

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